PHYC 581: High Energy Astrophysics
Fall 2014

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Aceretion of Plasma:

As we begin to consider the behavior of gases and plasmas under the influence of strong fields, a question that arises is whether a fluid description is applicable. A hydrodynamic approach is valid if the time scale for interactions between the constituent particles is shorter than that over which the external field changes the dynamics in the bulk.

The principal equations of hydrodynamics are formal representations of the conservation of mass, momentum, and energy. The first equation of hydrodynamics describes the conservation of the mass:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

By using the identity $\vec{\nabla}\cdot(\rho\vec{v}) = \rho(\vec{\nabla}\cdot\vec{v}) + \vec{v}\cdot(\vec{\nabla}\rho)$, we find

$$\frac{d\rho}{dt} + \rho(\vec{\nabla}\cdot\vec{v}) = 0 \Rightarrow \frac{1}{\rho}\frac{d\rho}{dt} = -\vec{\nabla}\cdot\vec{v}$$

Considering a mass element of volume δV, this implies that:

$$\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \vec{\nabla}\cdot\vec{v}$$

The second equation is the analogue of Newton's second law, and states the conservation of the momentum. The momentum of a mass element of volume δV is:

$$\vec{P} = \rho\vec{v}\,\delta V$$

Thus:

$$\frac{d\vec{P}}{dt} = \frac{d}{dt}(\rho\vec{v}\,\delta V) = \delta V\frac{d}{dt}(\rho\vec{v}) + \delta V\rho\vec{v}\frac{1}{\delta V}\frac{d(\delta V)}{dt} \Rightarrow$$

$$\frac{d\vec{P}}{dt} = \delta V\frac{d}{dt}(\rho\vec{v}) - \delta V\,\vec{v}\frac{d\rho}{dt}$$

On the other hand:

$$\frac{d\vec{P}}{dt} = \vec{F} = \vec{F_b} - \vec{\nabla}(\sigma_{ij})\,\delta V$$

Here $\vec{F_b}$ is the "body" force from an external field (like gravity), and σ_{ij} is the energy-momentum tensor:

$\sigma_{ij} = \rho v_i v_j$

It represents the i-th component of the momentum density flux in the j-th direction. The energy momentum tensor can be decomposed into a diagonal part and an off-diagonal part:

$\sigma_{ij} = P\delta_{ij} + \sigma_{0,ij}$

Here P is the pressure.

Using the identity $\vec{\nabla}\cdot(\rho v_i \vec{v}) = \rho v_i (\vec{\nabla}\cdot\vec{v}) + \vec{v}\cdot(\vec{\nabla}\rho v_i)$,

and the conservation of mass, we find:

$$\frac{dP_i}{dt} = \delta V\left[\frac{\partial(\rho v_i)}{\partial t} + \vec{v}\cdot(\vec{\nabla}\rho v_i) - v_i\frac{d\rho}{dt}\right] =$$

$$\delta V\left[\frac{\partial}{\partial t}(\rho v_i) + \vec{v}\cdot(\vec{\nabla}\rho v_i) - \rho v_i(\vec{\nabla}\cdot\vec{v})\right] =$$

$$\delta V\left[\frac{\partial}{\partial t}(\rho v_i) + \vec{\nabla}\cdot(\rho v_i \vec{v})\right]$$

But:

$$\frac{dP_i}{dt} = F_i = F_{b,i} - \delta V\frac{\partial \sigma_{ij}}{\partial x_j}$$

We then arrive at the second equation of hydrodynamics:

$$\frac{\partial}{\partial t}(\rho v_i) + \vec{\nabla}\cdot(\rho v_i \vec{v}) = -\frac{\partial \sigma_{ij}}{\partial x^j} + \frac{1}{\delta V} F_{b,i}$$

Finally, the third equation of hydrodynamics represents the conservation of the total energy. Let us start with the kinetic energy of a mass element $\frac{1}{2}\rho v^2 \delta V$. We have:

$$\vec{v}\cdot\frac{d}{dt}(\rho\vec{v}\,\delta V) = \vec{v}\cdot\vec{F}$$

It can be shown that:

$$\vec{v}\cdot\frac{d}{dt}(\rho\vec{v}) = \frac{d}{dt}\left(\frac{1}{2}\rho v^2\right) - \frac{1}{2}\rho v^2(\vec{\nabla}\cdot\vec{v})$$

Also:

$$\rho\vec{v}\,\frac{d\delta V}{dt} = \rho\vec{v}(\vec{\nabla}\cdot\vec{v})\,\delta V$$

Hence:

$$\vec{v}\cdot\frac{d}{dt}(\rho\vec{v}\,\delta V) = \delta V\left[\frac{d}{dt}\left(\frac{1}{2}\rho v^2\right) + \frac{1}{2}\rho v^2(\vec{\nabla}\cdot\vec{v})\right]$$

Neglecting the off-diagonal part of σ_{ij}, and for a body force from gravity $\vec{F_b} = -\rho \delta V \vec{\nabla} \Phi_g$ (where Φ_g is the gravitational potential), we have:

$$\vec{v} \cdot \vec{F} = \left[-\vec{v} \cdot \vec{\nabla} P - \rho (\vec{v} \cdot \vec{\nabla} \Phi_g) \right] \delta V$$

Thus:

$$\frac{d}{dt}\left(\frac{1}{2}\rho v^2\right) + \frac{1}{2}\rho v^2 (\vec{\nabla} \cdot \vec{v}) = -\vec{v} \cdot \vec{\nabla} P - \rho (\vec{v} \cdot \vec{\nabla} \Phi_g) \Rightarrow$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \vec{v} \cdot \left(\vec{\nabla} \frac{1}{2}\rho v^2\right) + \frac{1}{2}\rho v^2 (\vec{\nabla} \cdot \vec{v}) = -\vec{v} \cdot \vec{\nabla} P$$

$$-\rho(\vec{v} \cdot \vec{\nabla} \Phi_g) \Rightarrow \frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) + \vec{\nabla} \cdot \left(\frac{1}{2}\rho v^2 \vec{v}\right) =$$

$$-\vec{v} \cdot \vec{\nabla} P - \rho (\vec{v} \cdot \vec{\nabla} \Phi_g)$$

We note that:

$$-\rho(\vec{v} \cdot \vec{\nabla} \Phi_g) = -\vec{\nabla} \cdot (\rho \vec{v} \Phi_g) - \frac{\partial}{\partial t}(\rho \Phi_g) + \rho \frac{\partial \Phi_g}{\partial t}$$

In most situations $\frac{\partial \Phi_g}{\partial t} \simeq 0$. This implies that:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \Phi_g\right) + \vec{\nabla} \cdot \left[\left(\frac{1}{2}\rho v^2 + \rho \Phi_g\right)\vec{v}\right] = -\vec{v} \cdot \vec{\nabla} P$$

Next, we consider the internal energy of a mass element of volume δV. It is given by $\rho u \delta V$, where u is the internal energy per unit mass. From the first law of thermodynamics, we have:

$$\frac{dU}{dt} = T\frac{ds}{dt} - P\frac{d\delta V}{dt}$$

Where s is the entropy per unit mass. We note that

$\frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{-1}{\rho}\frac{d\rho}{dt}$. Hence:

$$\frac{dU}{dt} = T\frac{ds}{dt} + \frac{P}{\rho^2}\frac{d\rho}{dt}$$

Now:

$$\frac{d}{dt}(\rho U) = U\frac{d\rho}{dt} + \rho\frac{dU}{dt} = \rho T\frac{ds}{dt} + \left(U + \frac{P}{\rho}\right)\frac{d\rho}{dt}$$

Since $\frac{1}{\rho}\frac{d\rho}{dt} = -\vec{\nabla}\cdot\vec{V}$, we have:

$$\frac{d}{dt}(\rho U) + \rho U(\vec{\nabla}\cdot\vec{V}) = \rho T\frac{ds}{dt} - P(\vec{\nabla}\cdot\vec{V})$$

$$\frac{d}{dt}(\rho U) = \frac{\partial}{\partial t}(\rho U) + \vec{V}\cdot(\vec{\nabla}\rho U)$$

Thus,

$$\frac{d}{dt}(\rho U) + \rho U(\vec{\nabla}\cdot\vec{V}) = \frac{\partial}{\partial t}(\rho U) + \vec{\nabla}\cdot(\rho U) \Rightarrow$$

$$\frac{\partial}{\partial t}(\rho U) + \vec{\nabla}\cdot(\rho U\vec{V}) = \rho T\frac{ds}{dt} - P(\vec{\nabla}\cdot\vec{V})$$

Combined with the equation for the kinetic energy and gravitational potential energy (at the bottom of page 5), we arrive at:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho \Phi_g + \rho u\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho v^2 + \rho\Phi_g + \rho u\right)\vec{v}\right] = -\vec{v}\cdot\vec{\nabla}P - P(\vec{\nabla}\cdot\vec{v}) + \rho T\frac{ds}{dt}$$

The third equation of hydrodynamics when $\sigma'_{ij} \approx 0$ takes the following form:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\Phi_g + \rho u\right) + \vec{\nabla}\cdot\left[\left(\frac{1}{2}\rho v^2 + \rho\Phi_g + \rho u\right)\vec{v}\right] = \rho T\frac{ds}{dt} - \vec{\nabla}\cdot(P\vec{v})$$

With the help of the three hydrodynamic equations, we will discuss accretion of plasma onto compact objects in the next lecture.